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# Necessary Conditions for Singular Extremals

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**The purpose of this paper is to derive a set of necessary conditions for singular arcs. In this case, the classical Weierstrass and Legendre tests as well as the maximum principle fail to determine the nature of the extremal path. In the present analysis the second variation of the function to be minimized is evaluated for explicitly defined control variations. The dominant term of a power series in $\tau$, a parameter of the control variation which is allowed to approach zero in the limit, is calculated and examined for semidefiniteness. Should this term be zero for a particular problem, a new control variation is chosen and the procedure repeated. These control variations all belong to a special class of functions that were constructed with the satisfaction of terminal boundary conditions in mind. These boundary conditions are satisfied by applying secondary control variations that contribute terms to the second variation that are at least one degree higher in $\tau$. The results of the analysis are applied to the specific problem of rocket flight in an inverse square law field. It is shown that, for the time open case, the singular arc (Lawden's spiral) is nonoptimal.**

## Introduction

SINGULAR extremals are usually associated with variational problems in which the control variables appear linearly in the system differential equations. A singular arc or subarc occurs when the pseudo-Hamiltonian function $H$ is not explicitly a function of the control variable over a nonzero interval of an extremal arc. When such a situation exists, neither the maximum principle nor the classical variational theory provides necessary conditions for the arc to be minimizing.

Several papers have discussed specific examples that exhibit singular arcs and methods of analysis.[1–5] The approach taken here is a rather general one that could be considered as an extension of Kelley's work.[3] The positive semidefiniteness of the second variation of the payoff function is examined for a special class of explicitly defined control variations. When the first member of this class is used as a control variation the results are identical with that presented in Ref. 3. However, if the inequality of the test is met marginally (equality), in which case the test is inconclusive on the nature of the extremal arc, the second special control variation in the class is used and again the positive semidefiniteness of the second variation examined. If this test should also be satisfied marginally, the third control variation may be used and so on.

In the final analysis the second variation of the payoff function is evaluated as a power series in $\tau$ for a control variation that is a general member of the class of special control variations.‡ The variable $\tau$ is a parameter of the control variation which is allowed to approach zero in the limit. Each successive term in the power series provides a sequence of necessary conditions, each successively being applicable when the former is identically equal to zero. The general term of the series is expressed recursively as a function of the previous term. The special class of control variations has been constructed so that terminal conditions can be satisfied by an additional control variation that contributes terms to the second variation which are at least one degree higher in $\tau$.

Although not apparent, it is shown in the appendix that the sequence of necessary conditions can be restated concisely as $(-1)^k(\partial/\partial u)(d^{2k}/dt^{2k})H_u \geq 0$. This expression is a generalization of the Legendre condition for nonsingular extremals and was first obtained independently in this form by Robbins. Although the expression given in the foregoing is a more concise statement of the sequence of necessary conditions derived in this paper, its derivation lacks some of the motivation of the former development and is thus delegated to the appendix. The actual application of the equivalent tests involves about the same amount of computation.

## Problem Formulation

The problem is formulated as a Mayer problem; that is, given a system of differential equations and specified boundary conditions

$$\dot{x}_i = f_i(x_1, \ldots, x_n, u_1, \ldots, u_r, t) \qquad i = 1, \ldots, n$$
$$x_i(0) = x_{i0} \qquad i = 1, \ldots, n \tag{1}$$
$$x_i(t_f) = x_{if} \qquad i = 1, \ldots, m \qquad m \leq n$$

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‡ It has been tacitly assumed in the analysis to follow that the power series in $\tau$ for the second variation of the payoff functional has a nonzero interval of convergence.

minimize a function $P(x_{m+1f}, \ldots, x_{nf}, t_f)$ of the open final state variables $x_{if}$. The minimization is usually subject to constraints on the control variables $u_l$ which are denoted by requiring the control vector $\mathbf{u}$ to belong to a class of admissible controls $U$. However, such constraints will offer no difficulty in this analysis since it will be assumed that the control $\bar{\mathbf{u}}$ corresponding to a singular subarc $\bar{\mathbf{x}}$ is interior to the boundary of $U$, as is usually the case.

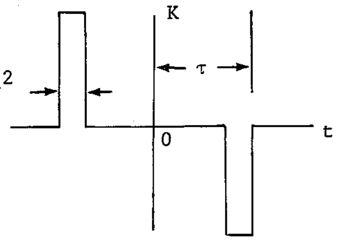


**Fig. 1** $\varphi_0{}^1(t, \tau)$.

Necessary conditions for $P$ to be a minimum are that the Pontryagin pseudo-Hamiltonian function $H$ be a minimum for all admissible controls satisfying the boundary conditions imposed on the system equations:

$$H(\bar{u}_1 + \Delta u_1, \ldots, \bar{u}_r + \Delta u_r) \geq H(\bar{u}_1, \ldots, \bar{u}_r) \quad (2)$$

where

$$H \equiv \sum_{i=1}^{n} \lambda_i f_i(x_1, \ldots, x_n, u_1, \ldots, u_r, t) \quad (3)$$

The $\lambda_i$ variables are the Lagrange multipliers and satisfy the differential equations

$$\dot{\lambda}_i = -\sum_{j=1}^{n} \lambda_j \frac{\partial f_j}{\partial x_i} \qquad i = 1, \ldots, n$$
$$\lambda_i(t_f) = \frac{\partial P}{\partial x_i} \qquad i = m+1, \ldots, n \quad (4)$$

Differential equations (1) and (4) can be put in the somewhat more convenient canonical form

$$\dot{x}_i = \partial H/\partial \lambda_i \qquad \dot{\lambda}_i = -\partial H/\partial x_i \qquad i = 1, \ldots, n \quad (5)$$

Singular subarcs occur when inequality (2) is met marginally (equality) during a nonzero interval of time for a variation of one of the control variables $u_l$ about an extremal arc. In the case where $u_l$ appears linearly in $H$, a singular subarc arises when the coefficient of the $u_l$ term $(\partial H/\partial u_l)$, which is usually referred to as the switching function, is identically equal to zero during a nonzero interval of time. In this case the maximum principle or the Weierstrass condition (2) fails to determine the nature of the extremal arc and fails to distinguish between controls that provide a minimum or a maximum over short lengths of arc.

Proceeding as in Ref. 1, the variation in $P$ due to a variation in control $\mathbf{u}$ is

$$\Delta P = \sum_{i=m+1}^{n} \frac{\partial P(\bar{\mathbf{x}}_f)}{\partial x_i} \Delta x_{if} + \frac{1}{2} \sum_{i,j=m+1}^{n} \frac{\partial^2 P(\bar{\mathbf{x}}_f + \xi \Delta \bar{\mathbf{x}}_f)}{\partial x_i \partial x_j} \Delta x_{if} \Delta x_{jf} =$$
$$\int_{t_0}^{t_f} \sum_{l=1}^{r} \frac{\partial \bar{H}}{\partial u_l} \Delta u_l dt + \int_{t_0}^{t_f} \sum_{\substack{i=1\\l=l}}^{\substack{n\\r}} \frac{\partial^2 \bar{H}}{\partial u_l \partial x_i} \Delta u_l \Delta x_i dt +$$
$$\frac{1}{2} \int_{t_0}^{t_f} \sum_{i,j=1}^{n} \frac{\partial^2 H(\mathbf{x} + \xi \Delta \mathbf{x}, \bar{\mathbf{u}} + \Delta \mathbf{u}, t)}{\partial x_i \partial x_j} \Delta x_i \Delta x_j dt +$$
$$\frac{1}{2} \sum_{i,j=m+1}^{n} \frac{\partial^2 P(\bar{\mathbf{x}}_f + \xi \Delta \bar{\mathbf{x}}_f)}{\partial x_i \partial x_j} \Delta x_{if} \Delta x_{jf} \quad 0 \leq \xi \leq 1 \quad (6)$$

A special class of control variations whose members are designated by $\varphi_0{}^q(t, \tau)$ will be used to evaluate (6) as a power series in $\tau$. These control variations have their support on the time interval $2\tau$ with a maximum value of $K$ and have been chosen such that terminal conditions can be satisfied without affecting the dominant term of the power series for $\Delta P$. The interval $2\tau$ can be any connected subinterval of the singular arc. In the course of the analysis, $K$ and $\tau$ will be allowed to approach zero independently. The first of these functions $\varphi_0{}^1(t, \tau)$ is very similar to the variation considered by Kelley and is shown in Fig. 1. The remaining members of the class of control variations and their construction will be discussed later. Successive integrations with respect to $t$ of $\varphi_0{}^q(t, \tau)$ are designated by $\varphi_v{}^q(t, \tau)$, that is,

$$d^v/dt^v \, \varphi_v{}^q(t, \tau) = \varphi_0{}^q(t, \tau) \quad (7)$$

The situation to be considered is that in which inequality (2) is met marginally (equality) for a variation of one of the control variables. The variation in $P$ to second-order terms in $K$ due to a variation in the singular control is calculated from (6). Since only a variation of the singular control is considered, the subscript on $u_l$ will be dropped in the remainder of this analysis. The second variation in the payoff function to second-order terms in $K$ which is designated by $\Delta P_2$ thus becomes from (6)

$$\Delta P_2 = \int_{t_0}^{t_f} \sum_{i=1}^{n} \frac{\partial^2 \bar{H}}{\partial u \partial x_i} \delta x_i \Delta u dt + \frac{1}{2} \int_{t_0}^{t_f} \sum_{i,j=1}^{n} \frac{\partial^2 \bar{H}}{\partial x_i \partial x_j} \times \delta x_i \delta x_j dt + \frac{1}{2} \sum_{i,j=m+1}^{n} \frac{\partial^2 \bar{P}}{\partial x_i \partial x_j} \delta x_{if} \delta x_{jf} \quad (8)$$

where

$$\delta \dot{x}_i = \sum_{j=1}^{n} \frac{\partial^2 \bar{H}}{\partial \lambda_i \partial x_j} \delta x_j + \frac{\partial^2 \bar{H}}{\partial \lambda_i \partial u} \Delta u$$
$$\delta x_i(t_0) = 0 \qquad i = 1, \ldots, n \quad (9)$$
$$\delta x_i(t_f) = 0 \qquad i = 1, \ldots, m, \; m < n$$

The terms in (8) involving the first and second partials of $H$ with respect to $u$ are identically zero along the singular arc.

Solving (9) for the case of a control variation $\Delta u = \varphi_0{}^q(t, \tau)$, under proper assumptions of smoothness gives

$$\delta x_i(t) = \sum_{v=1}^{q+1} A_{i,v}(t) \, \varphi_v{}^q(t, \tau) + \xi_i{}^q(t) \quad (10)$$

where

$$A_{i,1}(t) = \partial^2 \bar{H}/(\partial \lambda_i \partial u)$$
$$A_{i,v}(t) = \sum_{s=1}^{n} \frac{\partial^2 \bar{H}}{\partial \lambda_i \partial x_s} A_{s,v-1} - \frac{d}{dt}(A_{i,v-1}) \qquad v = 2, \ldots, q+1 \quad (11)$$
$$\xi_i{}^q = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} \xi_j{}^q + \left( \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} A_{j,q+1} - \dot{A}_{i,q+1} \right) \varphi_{q+1}{}^q$$
$$\xi_i{}^q(t_0) = 0$$

The first control variation to be considered is $\Delta u = \varphi_0{}^1(t, \tau)$. The time $t = 0$ is designated as the center of the interval $2\tau$ and may occur at any interior point of the singular arc ($\tau$ will be allowed to approach zero in the limit). From (8) and (10) $\Delta P_2$ becomes

$$\Delta P_2 = \int_{-\tau}^{\tau} \sum_{i=1}^{n} \frac{\partial^2 \bar{H}}{\partial u \partial x_i} \varphi_0{}^1(t, \tau) \left[ \sum_{v=1}^{2} A_{i,v}(t) \, \varphi_v{}^1(t, \tau) + \xi_i{}^1(t) \right] dt + \frac{1}{2} \int_{-\tau}^{t_f} \sum_{i,j=1}^{n} \frac{\partial^2 \bar{H}}{\partial x_i \partial x_j} \left[ \sum_{v=1}^{2} A_{i,v}(t) \varphi_v{}^1(t, \tau) + \xi_i{}^1(t) \right] \left[ \sum_{w=1}^{2} A_{j,w}(t) \varphi_w{}^1(t, \tau) + \xi_j{}^1(t) \right] dt +$$
$$\frac{1}{2} \sum_{i,j=m+1}^{n} \frac{\partial^2 \bar{P}}{\partial x_i \partial x_j} [A_{i,2}(t_f) \varphi_2{}^1(t_f, \tau) + \xi_i{}^1(t_f)] \times [A_{j,2}(t_f) \varphi_2{}^1(t_f, \tau) + \xi_j{}^1(t_f)] \quad (12)$$

and

$$\delta x_i(t) = \sum_{v=1}^{2} A_{i,v}(t)\, \varphi_v^1(t,\tau) + \xi_i^1(t) \tag{13}$$

The change $\Delta P_2$ in the payoff is easily evaluated as a power series in $\tau$ by integrating (12) by parts. Evaluating only the dominant terms gives

$$\Delta P_2 = \left\{ -\frac{1}{2}\frac{d}{dt}\left(\sum_{i=1}^{n} \frac{\partial^2 \bar{H}}{\partial u \partial x_1} A_{i,1}\right) - \sum_{i=1}^{n} \frac{\partial^2 \bar{H}}{\partial u \partial x_i} A_{i,2} + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 \bar{H}}{\partial x_i \partial x_j} A_{i,1} A_{j,1} \right\}\Bigg|_{t=0} K^2\tau^5 + 0(\tau^6) \tag{14}$$

A necessary condition for $P$ to be a minimum is that $\Delta P_2 \geq 0$. Since $t = 0$ can be any interior point on the singular arc, a necessary condition for the arc to be minimizing is

$$-\left\{ \frac{1}{2}\frac{d}{dt}\left(\sum_{i=1}^{n} \frac{\partial^2 \bar{H}}{\partial u \partial x_i} A_{i,1}\right) + \sum_{i=1}^{n} \frac{\partial^2 \bar{H}}{\partial u \partial x_i} A_{i,2} - \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 \bar{H}}{\partial x_i \partial x_j} A_{i,1} A_{j,1} \right\} \geq 0 \tag{15}$$

This is identical with the results obtained by Kelley[3] and was recognized by Bryson to be equivalent to the condition

$$\partial/\partial u[(d^2/dt^2)(\partial H/\partial u)] \leq 0 \tag{16}$$

At this point one is justified in being concerned with the admissibility of such a control variation in light of the boundary conditions imposed on (9). From (13) it is observed that

$$\delta x_i(t_f) = 2A_{i,2}(t_f)K\tau^3 + \xi_i^1(t_f) + 0(\tau^4) \tag{17}$$

where $\xi_i^1(t_f)$ is of order $\tau^3$. Therefore corrections in $\delta x_i(t_f)$ to satisfy boundary conditions can be made with control variations $\Delta u$ of order $\tau^3$ which contribute to variations in $P$ of order $\tau^6$. Thus the dominant term in $\Delta P_2$ is unchanged in (14). The existence of such variations is equivalent to a normality assumption.

If the equality is satisfied in (15), the second member $\varphi_0^2(t,\tau)$ (Fig. 2) of the class is used as a control variation and a similar procedure as before followed. After integrating by parts, $\Delta P_2$ for this variation becomes

$$\begin{aligned}\Delta P_2 = &\int_{-\tau}^{\tau} \left\{ -\frac{1}{2}\frac{d}{dt}\left(\sum_{i=1}^{n} \frac{\partial^2 \bar{H}}{\partial u \partial x_i} A_{i,1}\right) - \sum_{i=1}^{n} \frac{\partial^2 \bar{H}}{\partial u \partial x_i} A_{i,2} + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 \bar{H}}{\partial x_i \partial x_j} A_{i,1} A_{j,1} \right\} \times \\ &\{\varphi_1^2(t,\tau)\}^2\, dt + \int_{-\tau}^{\tau} \left\{ \frac{1}{2}\frac{d^2}{dt^2}\left(\sum_{i=1}^{n} \frac{\partial^2 \bar{H}}{\partial u \partial x_i} A_{i,2}\right) + \frac{3}{2}\frac{d}{dt}\left(\sum_{i=1}^{n} \frac{\partial^2 \bar{H}}{\partial u \partial x_i} A_{i,3}\right) + \sum_{i=1}^{n} \frac{\partial^2 \bar{H}}{\partial u \partial x_i} A_{i,4} - \right. \\ &\frac{1}{2}\frac{d}{dt}\left(\sum_{i,j=1}^{n} \frac{\partial^2 \bar{H}}{\partial x_i \partial x_j} A_{i,1} A_{j,2}\right) - \sum_{i,j=1}^{n} \frac{\partial^2 \bar{H}}{\partial x_i \partial x_j} A_{i,1} A_{j,3} + \left. \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 \bar{H}}{\partial x_i \partial x_j} A_{i,2} A_{j,2} \right\} \{\varphi_2^2(t,\tau)\}^2\, dt + \\ &\frac{1}{2}\sum_{i,j=m+1}^{n} \frac{\partial^2 \bar{P}}{\partial x_i \partial x_j} [A_{i,3}(t_f)\varphi_3^2(t_f,\tau) + \xi_i^2(t_f)] \times [A_{j,3}(t_f)\varphi_3^2(t_f,\tau) + \xi_j^2(t_f)] + \int_{-\tau}^{t_f} \beta(t,\tau)dt\end{aligned} \tag{18}$$

where $\beta(t,\tau)$ designates remaining terms of higher order in $\tau$. The coefficient of $\{\varphi_1^2(t,\tau)\}^2$ is identically equal to zero by assumption [Eq. (15)]. Evaluating (18) in terms of a power series in $\tau$ and keeping only dominant terms gives

$$\begin{aligned}\Delta P_2 = &\left\{ \frac{1}{2}\frac{d^2}{dt^2}\left(\sum_{i=1}^{n} \frac{\partial^2 \bar{H}}{\partial u \partial x_i} A_{i,2}\right) + \frac{3}{2}\frac{d}{dt}\left(\sum_{i=1}^{n} \frac{\partial^2 \bar{H}}{\partial u \partial x_i} A_{i,3}\right) + \sum_{i=1}^{n} \frac{\partial^2 \bar{H}}{\partial u \partial x_i} A_{i,4} - \right. \\ &\frac{1}{2}\frac{d}{dt}\left(\sum_{i,j=1}^{n} \frac{\partial^2 \bar{H}}{\partial x_i \partial x_j} A_{i,1} A_{j,2}\right) - \sum_{i,j=1}^{n} \frac{\partial^2 \bar{H}}{\partial x_i \partial x_j} A_{i,1} A_{j,3} + \\ &\left. \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 \bar{H}}{\partial x_i \partial x_j} A_{i,2} A_{j,2} \right\} K^2\tau^{11} + 0\,(\tau^{12}) \geq 0\end{aligned} \tag{19}$$

from which we obtain the second in a sequence of necessary conditions. The admissibility with regard to boundary constraints of such a control variation follows the same type of argument as before.

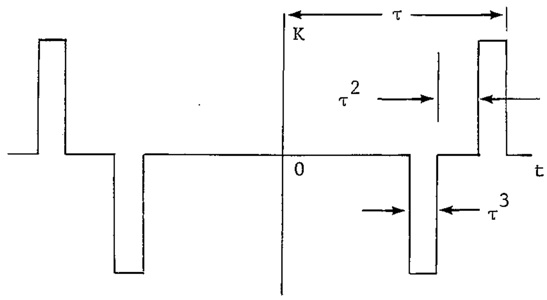


**Fig. 2** $\varphi_0^2\,(t,\tau)$.

If the equality condition is satisfied in (19) the third control variation of the class is used and so on. The motivation for choosing such a class of control variations arises from the theory of distributions. To find $\varphi_0^{q+1}(t,\tau)$ consider the derivative of $\varphi_0^q(t,\tau)$ accepting the Dirac Delta, and approximate this distribution by a pulse of width $\tau^{q+2}$ and scale so that the magnitude of the pulses is $K$.

## General Theory

From the previous discussion for the control variations $\varphi_0^1$ and $\varphi_0^2$, it becomes evident that it is not necessary to actually construct the specific control variation but only to be assured that such variations exist for which a constructive method has been given. With these thoughts in mind we proceed toward a general development.

With $\Delta u = \varphi_0^q(t,\tau)$, (8) is evaluated to second-order terms in $K$, giving

$$\begin{aligned}\Delta P_2 = &\int_{-\tau}^{\tau} \sum_{i=1}^{n} \frac{\partial^2 \bar{H}}{\partial u \partial x_i} \varphi_0^q(t,\tau) \left[ \sum_{v=1}^{q+1} A_{i,v}(t)\varphi_v^q(t,\tau) + \xi_i^q(t) \right] dt + \frac{1}{2}\int_{-\tau}^{t_f} \sum_{i,j=1}^{n} \frac{\partial^2 \bar{H}}{\partial x_i \partial x_j} \times \\ &\left[ \sum_{\substack{v=1 \\ w=1}}^{q+1} A_{i,v}(t)\varphi_v^q(t,\tau) + \xi_i^q(t) \right] \times [A_{j,w}(t)\varphi_w^q(t,\tau) + \xi_j^q(t)]dt + \frac{1}{2}\sum_{i,j=m+1}^{n} \frac{\partial^2 \bar{P}}{\partial x_i \partial x_j} \delta x_i(t_f)\delta x_j(t_f)\end{aligned} \tag{20}$$

Successive integrations by parts of (20) $q$ times gives

$$\Delta P_2 = \int_{-\tau}^{\tau} \left\{ -\sum_{i=1}^{n} \frac{\partial^2 \bar{H}}{\partial u \partial x_i} A_{i,2} - \frac{1}{2}\frac{d}{dt} \times \left(\frac{\partial^2 \bar{H}}{\partial u \partial x_i} A_{i,1}\right) + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 \bar{H}}{\partial x_i \partial x_j} A_{i,1} A_{j,1}\right\} \times \{\varphi_1{}^q(t,\tau)\}^2\, dt + \ldots + \int_{-\tau}^{\tau}\left\{\sum_{s=1}^{k} \eta_{sk} + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 \bar{H}}{\partial x_i \partial x_j} A_{i,k} A_{j,k}\right\}\{\varphi_k{}^q(t,\tau)\}^2\, dt + \ldots + \int_{-\tau}^{\tau}\left\{\sum_{s=1}^{q} \eta_{sq} + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 \bar{H}}{\partial x_i \partial x_j} A_{i,q} A_{j,q}\right\} \times \{\varphi_q{}^q(t,\tau)\}^2\, dt + \frac{1}{2}\sum_{i,j=m+1}^{n} \frac{\partial^2 \bar{P}}{\partial x_i \partial x_j} \times \delta x_i(t_f)\delta x_j(t_f) + \int_{-\tau}^{t_f} \beta(t,\tau)\, dt \quad (21)$$

where

$$\eta_{1k} = -\eta_{1,k-1}(A_{i,v+2}) - \frac{d}{dt}\{\eta_{1,k-1}(A_{i,v+1})\} \qquad k > 1$$

$$\eta_{sk} = -\eta_{s,k-1}(A_{i,v} A_{j,w+2}) - \frac{d}{dt} \times \{\eta_{s,k-1}(A_{i,v} A_{j,w+1})\} \qquad k > s > 1$$

$$\eta_{11} = -\left\{\sum_{i=1}^{n} \frac{\partial^2 H}{\partial u \partial x_i} A_{i,2} + \frac{1}{2}\frac{d}{dt}\frac{\partial^2 H}{\partial u \partial x_i} A_{i,1}\right\}$$

$$\eta_{ss} = -\left\{\sum_{i,j=1}^{n} \frac{\partial^2 H}{\partial x_i \partial x_j} A_{i,s-1} A_{j,s+1} + \frac{1}{2}\frac{d}{dt}\left(\frac{\partial^2 H}{\partial x_i \partial x_j} A_{i,s-1} A_{j,s}\right)\right\} \qquad s > 1$$

The function $\beta(t, \tau)$ represents all remaining terms arising from the integration by parts. From (13) the change in $x_i(t_f)$ due to the control variation $\varphi_0{}^q(t, \tau)$ becomes

$$\delta x_i(t_f) = A_{i,q+1}(t_f)\varphi_{q+1}{}^q(t_f, \tau) + \xi_i{}^q(t_f) \quad (22)$$

It can be shown that each successive integral in (21) is of higher order than the preceding one in the limit as $\tau$ approaches zero. If $q$ is chosen so that the first nonzero term is the $q$th term, then a necessary condition for $P_2$ to be a minimum is

$$\sum_{s=1}^{q} \eta_{sq} + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 H}{\partial x_i \partial x_j} A_{i,q} A_{j,q} \geq 0 \quad (23)$$

The quadratic term in $\delta x_i(t_f)$ will always be of one degree higher in $\tau$ than the terms arising out of (23), and thus can be neglected as $\tau$ approaches zero in the limit.

To satisfy boundary conditions on $x_i(t_f)$, control corrections are made in the remaining interval of time which will add to $\Delta P_2$ terms that are also of one degree higher in $\tau$ than those arising out of (23), and thus can be neglected as $\tau$ approaches zero in the limit.

Although not readily apparent it is shown in the appendix that the necessary condition given by (23) can be expressed equivalently as

$$(-1)^k(\partial/\partial u)[(d^{2k}/dt^{2k})(\partial H/\partial u)] \geq 0 \quad (24)$$

and was first derived independently by Robbins.

## Example

To illustrate the application of this test for singular extremals the problem of rocket flight in an inverse square law field for the time-open case will be considered. This problem has been analyzed extensively by Lawden[6,7] and others.[5] However, to the authors' knowledge, the nature of the singular arc (better known as Lawden's spiral) has been unresolved by these reports with one exception. Through personal correspondence, the authors find that Robbins[4] has by an independent investigation arrived at basically the same results we will show here, namely, that the singular arc is not a minimum fuel arc for the time-open case ($H = 0$).

The system equations are

$$\left.\begin{aligned} \dot{x}_1 &= -\mu x_3/R^3 + T/m \sin\Psi \\ \dot{x}_2 &= -\mu x_4/R^3 + T/m \cos\Psi \\ \dot{x}_3 &= x_1 \qquad \dot{x}_4 = x_2 \qquad \dot{x}_5 = \dot{m} = -T/C \end{aligned}\right\} \quad (25)$$

where

$$R = (x_3^2 + x_4^2)^{1/2}$$

and the problem is to choose $T$ and $\Psi$ such that $P = -m_f$ is minimized (minimum fuel) subject to the constraint $0 \leq T \leq T_{\max}$. The pseudo-Hamiltonian $H$ function becomes

$$H = \lambda_1\left(-\frac{\mu x_3}{R^3} + \frac{T}{m}\sin\Psi\right) + \lambda_2\left(-\frac{\mu x_4}{R^3} + \frac{T}{m}\cos\Psi\right) + \lambda_3 x_1 + \lambda_4 x_2 - \lambda_5\frac{T}{C} \quad (26)$$

where the $\lambda_i$ are the adjoint variables and obey the differential equations

$$\dot{\lambda}_i = -\partial H/\partial x_i \qquad i = 1, \ldots, 5 \quad (27)$$

From the classical theory $H$ is minimized with respect to $T$ and $\Psi$ giving

$$\sin\Psi = -\lambda_1/(\lambda_1^2 + \lambda_2^2)^{1/2} \qquad \cos\Psi = -\lambda_2/(\lambda_1^2 + \lambda_2^2)^{1/2}$$
$$T = T_{\max} \text{ when } \rho < 0 \qquad T = 0 \text{ when } \rho > 0 \quad (28)$$

where

$$\rho = \left[\frac{\lambda_1}{m}\sin\Psi + \frac{\lambda_2}{m}\cos\Psi - \frac{\lambda_5}{c}\right] = -\frac{(\lambda_1^2 + \lambda_2^2)^{1/2}}{m} - \frac{\lambda_5}{c} \quad (29)$$

The singular condition occurs when $\rho$ is identically equal to zero over a nonzero interval of time. The fact that $(\lambda_1^2 + \lambda_2^2)^{1/2}$ is then a constant derives from $(d/dt)(m\lambda_5) = 0$. Without loss of generality we set $(\lambda_1^2 + \lambda_2^2)^{1/2} = 1$.

Applying the first test given by (23) with $q = 1$ we obtain

$$\eta_{11} + \frac{1}{2}\sum_{i,j=1}^{5} \frac{\partial^2 H}{\partial x_i \partial x_j} A_{i,1} A_{j,1} \geq 0 \quad (30)$$

Substituting for $\eta_{11}$ this becomes

$$-\left\{\sum_{i=1}^{5} \frac{\partial^2 H}{\partial T \partial x_i} A_{i,2} + \frac{1}{2}\frac{d}{dt}\left(\sum_{i=1}^{5} \frac{\partial^2 H}{\partial T \partial x_i} A_{i,1}\right)\right\} + \frac{1}{2}\sum_{i,j=1}^{5} \frac{\partial^2 H}{\partial x_i \partial x_j} A_{i,1} A_{j,1} \geq 0 \quad (31)$$

The only nonzero terms will be contributed by $\partial^2 H/(\partial T \partial x_5) A_{5,1}$ and $\partial^2 H/\partial m^2\, A_{5,1}{}^2$ where

$$\frac{\partial^2 H}{\partial T \partial x_5} = -\left\{\frac{\lambda_1}{m^2}\sin\Psi + \frac{\lambda_2}{m^2}\cos\Psi\right\} = \frac{1}{m^2}$$

$$\frac{\partial^2 H}{\partial m^2} = \frac{2}{m^3} T[\lambda_1 \sin\Psi + \lambda_2 \cos\Psi] = -\frac{2T}{m^3} \quad (32)$$

$$A_{5,1} = -1/c$$

Evaluating (31) with these terms we find that it is satisfied with the equality sign.

Applying the second test, that is (23) with $q = 2$, we obtain

$$\eta_{1,2} + \eta_{2,2} + \frac{1}{2} \sum_{i,j=1}^{5} H_{x_i x_j} A_{i,2} A_{j,2} \geq 0 \tag{33}$$

where

$$\eta_{1,2} = \sum_{i=1}^{5} \frac{\partial^2 H}{\partial T \partial x_i} A_{i,4} + \frac{3}{2} \frac{d}{dt} \left( \sum_{i=1}^{5} \frac{\partial^2 H}{\partial T \partial x_i} A_{i,3} \right) + \frac{1}{2} \frac{d^2}{dt^2} \left( \sum_{i=1}^{5} \frac{\partial^2 H}{\partial T \partial x_i} A_{i,2} \right)$$

$$\eta_{2,2} = - \sum_{i,j=1}^{5} \frac{\partial^2 H}{\partial x_i \partial x_j} A_{i,1} A_{j,3} - \frac{1}{2} \frac{d}{dt} \left( \sum_{i,j=1}^{5} \frac{\partial^2 H}{\partial x_j \partial x_j} A_{i,1} A_{j,2} \right) \tag{34}$$

The evaluation of (34) shows $\eta_{1,2}$ and $\eta_{2,2}$ to be identically equal to zero. Further, the only terms that contribute to the sum in (33) are for $i$ and $j$ equal to 3 and 4. Without loss of generality we choose our coordinate system such that $x_3 = R$ and $x_4 = 0$ when the test is applied. The terms that contribute to (33) are

$$\frac{\partial^2 H}{\partial x_3^2} = -\frac{6\mu\lambda_1}{R^4} \qquad \frac{\partial^2 H}{\partial x_3 \partial x_4} = \frac{3\mu\lambda_2}{R^4} \qquad \frac{\partial^2 H}{\partial x_4^2} = \frac{3\mu\lambda_1}{R^4}$$

$$A_{3,2} = \frac{1}{m} \sin\Psi \quad A_{4,2} = \frac{1}{m}\cos\Psi \tag{35}$$

Substituting (35) into (33) and using (28) we obtain as a necessary condition

$$-(3\mu \sin\Psi/m^2R^4)\{3 - 5\sin^2\Psi\} \geq 0 \tag{36}$$

From Eqs. (5.94) and (5.105) of Ref. 7 we see that, along the singular arc for $H = 0$, $R$ and $T/m$ are given by

$$R = a \sin^6\varphi/(1 - 3\sin^2\varphi) \tag{37}$$

and

$$\frac{T}{m} = b \left( \frac{1 - 3\sin^2\varphi}{3 - 5\sin^2\varphi} \right) \frac{(27 - 75\sin^2\varphi + 60\sin^4\varphi)}{\sin^{11}\varphi} \tag{38}$$

where $a$ and $b$ are positive constants, and $\varphi$ is the angle between the thrust direction and the local horizontal. Note that $\varphi$ equals $\Psi$ for the position coordinates we have chosen. From (38) we see that $\sin\Psi$ must be positive and from (37) that

$$0 < \sin\Psi < (\tfrac{1}{3})^{1/2} \tag{39}$$

which violates condition (36), thus showing that the singular arc is not minimizing.

## Conclusions

Based on the theory of the second variation, a sequence of necessary conditions has been derived for the singular extremal. The positive semidefiniteness of the second variation of the payoff is examined for a special class of control variations. If the inequality of the test is met marginally (equality) for the first necessary condition, in which case the test is inconclusive on the nature of the extremal arc, the second test is applied and so on.

The special class of variations was constructed such that the change in the second variation of the payoff was dominated by that portion of the integral over which the control variation had its support. That is, the change in the payoff due to the primary control variation was of lower order than that due to the subsequent variation whose purpose was to restore the boundary constraints.

The application of the general theory to the specific problem of rocket flight in an inverse square field shows that the singular arc (Lawden's spiral) for the time-open case ($H = 0$) is nonminimizing.

## Appendix

An independent derivation of

$$(-1)^k \, \partial/\partial u[(d^{2k}/dt^{2k})H_u] \geq 0 \tag{A1}$$

as a necessary condition for a singular arc to be minimizing is given here. To second-order terms

$$\frac{d}{dt}\left(\sum_{i=1}^{n} \delta\lambda_i \delta x_i\right) = -\sum_{i,j=1}^{n} \bar{H}_{x_i x_j} \delta x_i \delta x_j - \sum_{i=1}^{n} \bar{H}_{x_i u} \delta x_i \delta u + \sum_{i=1}^{n} \bar{H}_{\lambda_i u} \delta\lambda_i \delta u \tag{A2}$$

where $\overline{(\ )}$ indicates evaluation of the term along the singular arc. Equation (A2) together with (8) gives

$$2\Delta P_2 = \int_{t_0}^{t_f} \sum_{i=1}^{n} (\bar{H}_{x_i u}\delta x_i + \bar{H}_{\lambda_i u}\delta\lambda_i)\,\delta u dt - \delta x_i \delta\lambda_i \Big|_{t_0}^{t_f} + \sum_{i,j=m+1}^{n} \bar{P}_{x_i x_j} \delta x_i \delta x_j \tag{A3}$$

However,

$$\delta H_u = \sum_{i=1}^{n} \bar{H}_{u\lambda_i}\,\delta x_i + \bar{H}_{u\lambda_i}\,\delta\lambda_i = H_u - \bar{H}_u = H_u \tag{A4}$$

Substituting (A4) into (A3) with

$$\delta u = \varphi_0{}^q(t, \tau) \tag{A5}$$

and integrating by parts $k$ times gives

$$2\Delta P = (-1)^k \int_{-\tau}^{\tau} \frac{d^k}{dt^k}(H_u)\varphi^q(t,\tau)dt - \sum_{i=1}^{n} \delta x_i \delta\lambda_i \Big|_{t_f} + \sum_{i,j=m+1}^{n} \bar{P}_{x_i x_j} \delta x_i \delta x_j \Big|_{t_f} \quad q > k \tag{A6}$$

It will now be assumed and proved shortly that $u$ first appears explicitly in an even order time derivative of $H_u$, that is

$$(d^{2k}/dt^{2k})H_u = (d^{2k}/dt^{2k})H_u(\mathbf{x}, \lambda, u, t) \tag{A7}$$

Expanding (A7) in a Taylor series about the singular arc for a control variation $\delta u = \varphi_0{}^q(t, \tau)$ gives

$$\frac{d^{2k}}{dt^{2k}} H_u = \frac{d^{2k}}{dt^{2k}}\bar{H}_u + \frac{\partial}{\partial u}\left(\frac{d^{2k}}{dt^{2k}}\bar{H}_u\right) \times \varphi_0{}^q(t,\tau) \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{d^{2k}}{dt^{2k}}\bar{H}_u\right)\delta x_i + \frac{\partial}{\partial \lambda_i}\left(\frac{d^{2k}}{dt^{2k}}\bar{H}_u\right)\delta\lambda_i + \ldots \tag{A8}$$

The integrand in (A6) can be evaluated using

$$\frac{d^k}{dt^k} H_u = \int_{-\tau}^{t} \cdots \int_{-\tau}^{\xi} \frac{d^{2k}}{dt^{2k}}(H_u)(d\xi)^k \tag{A9}$$

The dominant term in (A8) is $(\partial/\partial u)[(d^{2k}/(dt^{2k})\bar{H}_u]\varphi_0{}^q(t, \tau)$, and this is substituted for the integrand of (A9). The $(\partial/\partial u)[(d^{2k}/dt^{2k})\bar{H}_u]$ is expanded in a power series in time about $t = 0$, and the integration in (A9) is performed giving

$$\frac{d^k}{dt^k} H_u = \frac{\partial}{\partial u} \frac{d^{2k}}{dt^{2k}} \bar{H}_u \Big|_{t=0} \varphi_k{}^q(t, \tau) \tag{A10}$$

plus terms involving $\varphi_{k+1}{}^q(t, \tau)$, etc. Equation (A6) then becomes

$$2\Delta P = (-1)^k \int_{-\tau}^{\tau} \frac{\partial}{\partial u} \frac{d^{2k}}{dt^{2k}} H_u \bigg|_{t=0} \varphi_k{}^q(t, \tau)^2 \, dt \quad \text{(A11)}$$

plus terms of higher order in $\tau$. Therefore as $\tau$ approaches zero in the limit a necessary condition for the singular arc to be minimizing is

$$(-1)^k(\partial/\partial u)(d^{2k}/dt^{2k})H_u > 0 \quad \text{(A12)}$$

where $k$ is the smallest integer for which this function does not vanish.

It will now be shown that the first nonzero value of $(\partial/\partial u)[d^m/dt^m \, H_u]$ occurs for $m$ even. In the proof the explicit dependence of $H$ on time $t$ is not considered. However, this assumption is not restrictive since $t$ can always be eliminated through the definition of an additional state variable whose derivative is equal to unity:

$$(\partial/\partial u)[(d^m/dt^m)H_u] = -(\nabla H_u)^T S\nabla(d^{m-1}/dt^{m-1})H_u = (-1)^m(\nabla H_u)^T S\{\nabla(\nabla H)^T S\}^{m-1}(\Delta H_u) \quad \text{(A13)}$$

where

$$\nabla^T = \left[\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial \lambda_1}, \ldots, \frac{\partial}{\partial \lambda_n}\right]$$

and $S$ the $2n \times 2n$ square matrix

$$S = \begin{bmatrix} N & I \\ -I & N \end{bmatrix}$$

Expanding the $\nabla$ operator in the center brackets of (A13) gives

$$\frac{\partial}{\partial u}\left[\frac{d^m}{dt^m} H_u\right] = (-1)^m(\nabla H_u)^T \times S\left\{[\nabla(\nabla H)^T S] - I\frac{d}{dt}\right\}^{m-1}(\nabla H_u) \quad \text{(A14)}$$

Since $(\partial/\partial u)[(d^{m-1}/dt^{m-1})H_u]$ is identically equal to zero along the singular arc

$$\frac{d}{dt}\frac{\partial}{\partial u}\left[\frac{d^{m-1}}{dt^{m-1}} H_u\right] = -\frac{d}{dt}\left\{(\nabla H_u)^T S\nabla \frac{d^{m-2}}{dt^{m-2}} H_u\right\} =$$

$$-(\nabla H_u)^T S\left(I\overleftarrow{\frac{d}{dt}}\right)\nabla \frac{d^{m-2}}{dt^{m-2}} H_u - (\nabla H_u)^T S\nabla \frac{d^{m-1}}{dt^{m-1}} H_u -$$

$$(\nabla H_u)^T S\,\{\nabla(\nabla H)^T S\}\,\nabla \frac{d^{m-2}}{dt^{m-2}} H_u = 0 \quad \text{(A15)}$$

where $I(\overleftarrow{d/dt})$ designates that the derivative operation be performed on members to the left of the operator. From (A13) and (A15)

$$\frac{\partial}{\partial u}\left[\frac{d^m}{dt^m} H_u\right] = (\nabla H_u)^T \times S\left\{I\overleftarrow{\frac{d}{dt}} + [\nabla(\nabla H)^T S]\,\nabla \frac{d^{m-2}}{dt^{m-2}} H_u \quad \text{(A16)}\right.$$

Repeating the process $m - 1$ times, one obtains

$$\frac{\partial}{\partial u}\left[\frac{d^m}{dt^m} H_u\right] = (-1)^m(\nabla H_u)^T \times S\left\{I\overleftarrow{\frac{d}{dt}} + [\nabla(\nabla H)^T S]\right\}^{m-1}(\nabla H_u) \quad \text{(A17)}$$

which upon transposing gives

$$\frac{\partial}{\partial u}\left[\frac{d^m}{dt^m} H_u\right] = (\nabla H_u)^T \times S\left\{[\nabla(\nabla H)^T S] - I\frac{d}{dt}\right\}^{m-1}(\nabla H_u) \quad \text{(A18)}$$

Comparing (A18) and (A14) shows that $m$ must be even. The material of this Appendix is presented in Ref. 8 in a more detailed and slightly different form.

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